

18.100A MIDTERM EXAM

Thursday 22nd 2018, 9:35 - 10:50 am

Last Name :

First Name :

MIT email (if you have):

Instruction

As an open book exam, during the exam you can see

1. the textbook : *Introduction to Real Analysis* by A. Mattuck,
2. notes and copies (at most 500 sheets of paper).

However, the following are NOT allowed to use

1. electronic devices.
2. the other books except the textbook.

When you write the poofs of problems, you can cite Theorems, Properties, and Examples in the textbook Chapter 1-11.

However, you can not use exercises and problems in the textbook as well as problem sets, practice problems, and their solutions. If you have copies of the solutions and want to use them, please rewrite the proofs.

If your score is greater than 100, you will receive 100 points.

1. (20 points) Determine whether the following statements are true or false. If true then prove it, and if false then provide a counterexample.

(1) Suppose that $\{a_n\}$ is bounded and $a_n \neq -1$. Then, $\left\{\frac{1}{1+a_n}\right\}$ is also bounded.

(2) Suppose that $f(x)$ is defined for $x \approx 0$, and $xf(x)$ is continuous at 0. Then, $f(x)$ is also continuous at 0.

2. (25 points) Let $a_{n+1} = 2 + \sqrt{a_n}$ and $a_0 > 4$. Prove that the sequence $\{a_n\}$ is convergent, and the limit is 4.
(Hint: $x - \sqrt{x} - 2 = (\sqrt{x} - 2)(\sqrt{x} + 1)$.)

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3. (25 points) Suppose that S is a non-empty bounded set with $\inf S \geq 0$, and let S^2 be the set $S^2 = \{s^2 : s \in S\}$. Prove that $\sup S^2 = (\sup S)^2$.

4. (10 points) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} 2^n n^2 x^{2n}$, and explain why.

5. (20 points) Let $f(x)$ be an increasing function defined for $x \in (-\infty, +\infty)$. Suppose that given any two rational numbers $r, q \in \mathbb{Q}$, we have

$$|f(r) - f(q)| \leq |r - q|.$$

Prove that $f(x)$ is continuous on \mathbb{R} .

(You may need to use that fact that given any two different real numbers $x < y$, there exists a rational number r such that $x < r < y$.)

6.(10 points, bonus problem) Let $f(x)$ be defined for $x \in (-\infty, +\infty)$. Suppose that given any two real numbers $x, y \in \mathbb{R}$

$$(*) \quad tf(x) + (1-t)f(y) \geq f(tx + (1-t)y),$$

holds for all $t \in [0, 1]$. Prove that $f(x)$ is continuous on \mathbb{R} .

(We say $f(x)$ is a convex function if $(*)$ holds.)